Solutions for Meet 3

Round 1

1.Shade the double intersections and subtract out the triple intersection.

2. Consider four cases: where the first and last digits are even/odd respectively, odd/even, odd/odd, even/even. For the first three cases: count the number of choices for the first digit, and then the last digit, then the middle digit, then the remaining digits; for the last case: consider the final digit, then the first digit, then the middle, then the others. Number of choices:

$$3 \quad 7 \quad 4 \quad 6 \quad 3 = 1512$$
$$2 \quad 7 \quad 4 \quad 6 \quad 2 = 672$$
$$2 \quad 7 \quad 3 \quad 6 \quad 2 = 504$$
$$2 \quad 7 \quad 5 \quad 6 \quad 2 = 840$$

Total = 3528

3. Consider a three by three matrix, there are 9 squares with sides of length one, 4 squares with sides of length two, 1 with sides of length three, 4 with sides of lengths sqrt(2), and 2 with sides along diagonal of length sqrt(5), e.g., (0,1) to (1,3), for a total of 20.

For those wishing to pursue this new idea, consider the arrangement of numbers

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    9
   4 4
  1 1 1
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The numbers add to 20, as above.

When we change to a 4 by 4 matrix, the number of squares can be found

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     16
    9  9
   4  4  4
  1  1  1  1
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for a total of 50. (But the answer to the question asked is 20.)

4. $(a)\frac{2}{3}\cdot\frac{1}{3}=\frac{2}{9}$ $\quad (b)=\frac{1}{3}\cdot\frac{2}{3}=\frac{2}{9}$ $\quad (c)=\frac{2}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{1}{3}=\frac{5}{9}$. Answer $\left(\frac{2}{9},\frac{2}{9},\frac{5}{9}\right)$

5. Answer: 113. Only 1, 3, 7, 9 are possible digits. (a) If no digits are the same, they must be 1, 3, 7 (since 91 and 93 are composites), 7 divides 371. (b) If two digits are the same, they must be 1, 1. Then only 113 works (91 and 117 are composites, 131, 311 are primes)

Solutions for Meet 3

Round 2

1.One can analyze $\frac{1}{2}ab = a + b + \sqrt{a^2 + b^2}$ and reach $(a-4)(b-4) = 8$; this produces only $(5,12) and\ (6,8)$ as answers

2. Draw a diameter from the top of the semicircle and through the rectangle. The radius of the semicircle is 15, the width of the rectangle is 10, the length can be written as 15 and 15. The last part of the diameter will be x. So $x(15+10) = 15 \cdot 15, \quad so\ x = 9$. Then the diameter = 34 and the radius = 17

3. We can prove that $\Delta A'B'C' \cong \Delta ABC$, with corresponding sides parallel, $[Note\ that\ since\ sides\ BB'\ and\ CC'\ \ bisect\ each\ other, B'C'BC\ is\ a\ parallelogram]$ and let K be the midpoint of $B'C'$ $[Prove\ that\ \Delta B'KG \cong \Delta BMG]$. Then $A'G = AG = 2 \cdot GM$, so $A'M = GM = \frac{1}{3}AM = \frac{1}{3}A'K$. $BC \parallel B'C'$ makes triangle $A'PQ \sim \Delta A'B'C'$, with ratio of similitude 1:3, so area of $A'PQ = \frac{1}{9}\ the\ area\ A'B'C'$. Similarly the areas of $\Delta C'RS$ and $\Delta\ B'TW$ are each $\frac{1}{9}\ the\ area\ of\ \Delta A'B'C'$, so that the area of hexagon $PQRSTW\ is\ \frac{2}{3}\ the\ area\ of\ \Delta A'B'C'\ thus$ equaling 48.

4. The common roots must be roots of (subtract the equations) $2x^2 + (r-q) = 0$, so their sum is 0. Then the third root of the first equation must be -5 (since its roots add up to -5), and that of the second equation must be -7. Answer (-5, -7)

5. The intersections must lie on (add the equations) $2x^2 + 12x - 8y + 26 = 0$, which is the parabola $(x+3)^2 = 4(y-1)$. This parabola has vertex (-3, 1), focus (-3, 2), and directrix y = 0. Thus the sum of the distances to (-3, 2) equals the sum of the distances to the x-axis, which is

the sum of the ordinates of the intersection points. Those points must also satisfy (subtract the original equations) $y^2 - 20y + 59 = 0$, so the sum of two of the ordinates is 20, and the sum of all four (this is a hyperbola crossing a circle, producing pairs of points with the same ordinates) is 40.