Lecture Notes: Introduction to Algebra: Polynomials

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1. Background Information

Algebra is "the study of calculations on some set." In math team, algebra usually refers to the manipulation of mathematical expressions and equations. One of the most important objects in algebra is the polynomial.

A *polynomial* is a function in one or more variables that consists of a sum of variables raised to nonnegative, integral powers and multiplied by real coefficients. For this lecture, we will only use polynomials in one variable.

Let $P(x)$ be a polynomial in x. The highest power that x is raised to is called P's *degree*. If $P(m) = 0$ for some number m, m is called a *root* or a *zero* of P. Every polynomial P can be expressed as $a(x - r_1)(x - r_2)(x - r_3)...(x - r_n)$, where $r_1, r_2, ..., r_n$ are the roots of P.

2. Fundamental Theorem of Algebra

The foundational principle of polynomials and how they work is known as the *Fundamental Theorem of Algebra*. It states that for every positive integer n, a polynomial of degree n has n roots.

What about $7x^2 + 7x + 7$?

3. Vieta's Theorem

Consider any cubic polynomial $P(x)$. $P(x)$ is of the form $a(x - r)(x - s)(x - t)$, where r, s, and t are the roots of P.

Expanding, we get

$$P(x) = ax^3 - a(r + s + t)x^2 + a(rs + st + rt)x - a \cdot rst.$$

What we get is that the x^2-coefficient of $P(x)$ is the $-a(r+s+t)$, where a is the leading coefficient. Similarly, we have that the x^1-coefficient is $a(rs + st + rt)$ and that the x^0-coefficient is $-a \cdot rst$.

Here's why this is cool: let's say that $P(x) = 2x^3 + 5x^2 - 7x - 6$. We might not know the roots of $P(x)$, but we do know things *about* the roots. If the roots of $P(x)$ are r, s, and t, we know that $P(x) = 2(x - r)(x - s)(x - t)$. Expanding out and setting terms equal to each other, we get that $-2(r + s + t) = 5$, $2(rs + st + rt) = -7$, and $-2rst = -6$. Therefore, *the sum of the roots of* $P(x)$ *is* $r + s + t = -\frac{-a(r+s+t)}{a} = -\frac{5}{2}$, *the product of the roots is* 3, *and the pairwise sum of the roots is* $-\frac{7}{2}$.

This works for more than just cubic polynomials. If the roots of a quartic polynomial are r_1, r_2, r_3, and r_4, we have $P(x) = a(x - r_1)(x - r_2)(x - r_3)(x - r_4)$, where a is the leading coefficient of $P(x)$. We can expand this out and derive some identities. For example, if $P(x) = ax^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, then the sum of the roots of $P(x)$ is $-\frac{a_3}{a}$ and the product of the roots is $\frac{a_0}{a}$.

In general, for a monic degree n polynomial, the sum of the roots is the negative of the $n - 1$ coefficient, and the product of the roots is the constant coefficient for even n and the negative of the constant coefficient for odd n.

Can you find any seeming exceptions?

WARNING! When using Vieta's theorem, be extremely careful with your + and − signs! Also, don't forget to divide by the leading coefficient.

4. Rational Root Theorem

Theorem 4.1. *All rational roots of* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$*, where* a_i *is an integer for all* $1 \le i \le n$ *and* $a_n \neq 0$*, are of the form* $\pm\frac{\text{Factor of } a_0}{\text{Factor of } a_n}$.

Proof. Let $\frac{p}{q}$ be a root of $P(x)$, with p and q relatively prime (i.e. their greatest common divisor is 1). Then we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\frac{p}{q} + a_0 = 0.$$

Multiplying by q^n, we get

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = 0. \tag{1}$$

Thus,

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} = -a_0 q^n.$$

Taking a common factor of p out from the left-hand side of the equation, we get

$$p(a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}) = -a_0 q^n.$$

This means that $a_0 q^n$ is divisible by p. However, since p and q are relatively prime (so p and q^n are relative prime), we have that p is a divisor of a_0.

Now we carry $a_n p^n$ to the right-hand side of Equation 1. We get

$$a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = -a_n p^n.$$

Taking a common factor of q out this time, we get

$$q(a_{n-1}p^{n-1} + \cdots + a_1 pq^{n-2} + a_0 q^{n-1}) = -a_n p^n.$$

Thus, $a_n p^n$ is divisible by q, which, by the same logic as above, implies that q is a factor of a_n.

Therefore, the root $\frac{p}{q}$ is of the form $\frac{\pm\text{Factor of } a_0}{\text{Factor of } a_n}$, as desired.

Law 4.2. *In competition problems, you often need to find the rational roots of a polynomial. These roots are frequently* ± 1*. This is known as the* Law of Ones*.*

□

5. How to Find Roots

Theorem 5.1 (Descartes's Rule of Signs)**.** *The maximum number of positive roots of P(x) is the number of times consecutive coefficients change signs in P(x). The maximum number of negative roots of P(x) is the number of times consecutive coefficients change signs in P(-x).*

Example: $P(x) = -2x^5 - 10x^4 - 92x^3 - 154x^2 + 2454x - 2196$.

Theorem 5.2 (Intermediate Value Theorem for Polynomials)**.** *Given any arbitrary polynomial P(x), suppose that* $P(a) = c$ *and* $P(b) = d$*. Then, for every f between c and d, there exists a g between a and b such that* $P(g) = f$*.*

Why is this useful for computing roots?

Synthetic Division allows one to divide a polynomial by a binomial quickly while simultaneously checking if a number is a root.

Example on board.

Conjugate Pairs For a polynomial with rational coefficients, if $a + \sqrt{b}$ is a root, then $a - \sqrt{b}$ is also a root, and vice versa.

For a polynomial with real coefficients, if $a + bi$ is a root, then $a - bi$ is also a root, and vice versa.

Cool Coefficient-Flipping Fact (not actually useful for finding roots) If the order of coefficients is flipped, the roots are reciprocated.