

Problem Set: Introduction to Algebra: Polynomials

Montgomery Blair Math Team

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1. What is the sum of the roots of $-3x^4 + 5x^3 + 6x^2 - 7x + 2$? The product? (Hint: be extremely careful with your signs.)
2. Suppose that the sum of the roots of $P(x)$, a 3rd degree polynomial is 6. Find the sum of the roots of $P(x + 1)$.
3. Find all roots of $2x^5 + 7x^4 - 17x^3 - 7x^2 + 15x$.
4. $x^4 + 39x^3 + 394x^2 + 599x + 23$ has exactly one rational root. Find this root (without performing a large amount of computation).
5. Find the sum of the squares of the roots of $2x^3 + 9x^2 - 5x - 1$.
6. Let a , b , and c be real numbers such that $a - b + c = -17$, $ac - ab - bc = 54$, and $abc = -72$. Find all possible triples (a, b, c) .
7. Find the sum of the cubes of the roots of $x^3 + 3x + 61$.
8. Let r , s , and t be the roots of $x^3 + 5x^2 - 7x - 9$.

- a. Find $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$.
- b. Find the polynomial with leading coefficient -9 and roots $\frac{1}{r}$, $\frac{1}{s}$, and $\frac{1}{t}$.

9. Find a monic cubic polynomial whose roots satisfy the following properties:

- their average is 1
- the average of their squares is 2
- the average of their cubes is 5

10. This problem is not directly related to what we've covered today, but it is an interesting one nonetheless:

- a. Find the sum of the coefficients of $(x+1)(x-2)^{10}(x-3)^2(x-7)$.
- b. Find the sum of the odd-exponent coefficients of $(x+1)(x-2)^{10}(x-3)^2(x-7)$ (i.e. the sum of the coefficients of x^1 , x^3 , etc.).

11. The real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ can be written in the form $\frac{\sqrt[3]{a} + \sqrt[3]{b+1}}{c}$, where a , b , and c are positive integers. Find $a + b + c$. (AIME I 2013).

12. Suppose a , b are nonzero integers such that two roots of $x^3 + ax^2 + bx + 9a$ coincide, and all three roots are integers. Find $|ab|$. (PUMaC 2013).

13. Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Prove that $\frac{a^2 + b^2}{ab + 1}$ is a perfect square. (IMO 1988).