Solutions: Introduction to Algebra

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1. A direct application of Vieta's Theorem tells us that the sum of the roots is $\boxed{\dfrac{5}{3}}$ and the product of the roots is $\boxed{-\dfrac{2}{3}}$.

2. **Solution 1:** Without Loss Of Generality, let the leading coefficient be 1. By Vieta's, the coefficient of x^2 is -6, so $P(x) = x^3 - 6x^2 + Q(x)$, where $Q(x)$ has degree at most 1. Then, $P(x+1) = (x+1)^3 - 6(x+1)^2 + Q(x+1) = x^3 + (3x^2 - 6x^2) + (3x - 12x) + (1 - 6 + Q(x+1))$. $Q(x+1)$ does not have an x^2 term, so the x^2 coefficient in $P(x+1)$ is -3. By Vieta's again, the sum of the roots is $\boxed{3}$.

   Solution 2: Graphically, $P(x+1)$ is $P(x)$ shifted left by 1 unit, so each root of $P(x+1)$ is 1 less than the corresponding root in $P(x)$. By the Fundamental Theorem of Algebra, there are 3 roots, so the new sum is $6 - 3 = \boxed{3}$.

3. We use the Law of Ones, and guess $-1, 1$ as two of the roots. Luckily, -1 and 1 both happen to be roots of $2x^5 + 7x^4 - 17x^3 - 7x^2 + 15x$. Therefore, we know that $(x-1)(x+1) = x^2 - 1$ is a factor of $2x^5 + 7x^4 - 17x^3 - 7x^2 + 15x$, and we can use polynomial division or synthetic division to know that the remaining roots are roots of $2x^3 + 7x^2 - 15x$. 0 is also clearly a root since there is no constant term, so the remaining two roots must be roots of $2x^2 + 7x - 15$. We apply our factoring skills to factor this into $(x+5)(2x-3)$, so the roots are $\boxed{-5, -1, 0, 1, \dfrac{3}{2}}$

4. We try the Law of Ones, but unfortunately it does not work for this problem. Next, we try the rational root theorem. The only possible rational roots are $-23, -1, 0, 1, 23$. We know that $-1, 1$ don't work, 0 clearly does not work, and 23 does not work by the rule of signs. Thus, the rational root must be $\boxed{-23}$.

5. We wish to express $r_1^3 + r_2^3 + r_3^3$ in terms of the coefficients of the polynomial. One can show that:

$$r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_2r_3 + r_1r_3)$$

   Applying Vieta's, we have:

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 &= (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_2r_3 + r_1r_3) \\ &= \left(-\frac{9}{2}\right)^2 - 2\left(-\frac{5}{2}\right) \\ &= \frac{81}{4} + 5 \\ &= \boxed{\frac{101}{4}} \end{aligned}$$

6. By Vieta's Theorem, a polynomial with roots a, $-b$, and c is $x^3 + 17x^2 + 54x - 72$. The Law of Ones suggests trying 1 as a root, and indeed it is. Factoring out $x - 1$, we get $x^2 + 18x + 72$, the roots of which are -6 and -12. Thus, a, $-b$, and c are 1, -6, and -12 in some order. Thus, all values of (a, b, c) are:

$$\boxed{(1, 6, -12), (1, 12, -6), (-6, -1, -12), (-6, 12, 1), (-12, -1, -6), (-12, 6, 1)}.$$

7. Because all of the roots satisfy $x^3 + 3x + 61 = 0$, they all satisfy $x^3 = -3x - 61$. So,

$$\begin{aligned}\sum_{roots} r^3 &= \sum_{roots} (-3r - 61) \\ &= \sum_{roots} -3r + \sum_{roots} (-61)\end{aligned}$$

By Vieta's, $\sum_{roots} r = 0$, and by the Fundamental Theorem, there are 3 roots. So,

$$\begin{aligned}\sum_{roots} r^3 &= \sum_{roots} -3r + \sum_{roots} (-61) \\ &= \sum_{roots} (-61) \\ &= \boxed{-183}.\end{aligned}$$

8. a. We begin by rearranging fractions so that $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{rs+rt+st}{rst}$. Then, we simply apply Vieta's Theorem to get $\boxed{-\dfrac{7}{9}}$.

   b. Let us do this coefficient by coefficient. The leading coefficient is -9. The next coefficient is $-9(-(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}))$, which is -7. The next coefficient is $-9(\frac{1}{rs} + \frac{1}{st} + \frac{1}{rt}) = -9(\frac{r+s+t}{rst}) = -9(\frac{-5}{9}) = 5$. The last coefficient is $-9(-\frac{1}{rst}) = -9(-\frac{1}{9}) = 1$. Therefore, the polynomial is $\boxed{-9x^3 - 7x^2 + 5x + 1}$. (This process is the reason behind the coefficient-reversing trick.)

9. In order to determine the polynomial, we must find $r_1 + r_2 + r_3$, $r_1r_2 + r_1r_3 + r_2r_3$, and $r_1r_2r_3$. $r_1 + r_2 + r_3 = 3$. Squaring this, we get

$$r_1^2 + r_2^2 + r_3^2 + 2(r_1r_2 + r_1r_3 + r_2r_3) = 9$$

Subtracting the known equation

$$r_1^2 + r_2^2 + r_3^2 = 6$$

gives

$$\begin{aligned}2(r_1r_2 + r_1r_3 + r_2r_3) &= 3 \\ r_1r_2 + r_1r_3 + r_2r_3 &= \frac{3}{2}.\end{aligned}$$

Lastly, cubing $r_1 + r_2 + r_3 = 3$ gives

$$\begin{aligned}r_1^3 + r_2^3 + r_3^3 + 3(r_1^2r_2 + r_1^2r_3 + r_2^2r_1 + r_2^2r_3 + r_3^2r_1 + r_3^2r_2) + 6r_1r_2r_3 &= 27 \\ 3(r_1^2r_2 + r_1^2r_3 + r_2^2r_1 + r_2^2r_3 + r_3^2r_1 + r_3^2r_2) + 6r_1r_2r_3 &= 12\end{aligned}$$

Meanwhile, multiplying $r_1 + r_2 + r_3 = 3$ and $r_1^2 + r_2^2 + r_3^2 = 6$ and subtraction from the previous equation gives

$$\begin{aligned}r_1^3 + r_2^3 + r_3^3 + r_1^2r_2 + r_1^2r_3 + r_2^2r_1 + r_2^2r_3 + r_3^2r_1 + r_3^2r_2 &= 18 \\ (r_1^2r_2 + r_1^2r_3 + r_2^2r_1 + r_2^2r_3 + r_3^2r_1 + r_3^2r_2) &= 3 \\ 3(r_1^2r_2 + r_1^2r_3 + r_2^2r_1 + r_2^2r_3 + r_3^2r_1 + r_3^2r_2) &= 9 \\ 6r_1r_2r_3 &= 3 \\ r_1r_2r_3 &= \frac{1}{2}\end{aligned}$$

So, the final polynomial is $\boxed{x^3 - 3x^2 + \dfrac{3}{2}x - \dfrac{1}{2}}$.

10. a. Let $f(x) = (x+1)(x-2)^{10}(x-3)^2(x-7)$. $f(1)$ gives us the sum of the coefficients, and it equals $2(-1)^{10}(-2)^2(-6) = \boxed{-48}$

b. Let $f(x) = (x+1)(x-2)^{10}(x-3)^2(x-7)$. $\frac{f(1)-f(-1)}{2}$ gives us the sum of the odd-exponent coefficients, since $f(-1)$ gives us the sum of the even coefficients minus the sum of the odd coefficients. $\frac{f(1)-f(-1)}{2} = \frac{2(-1)^{10}(-2)^2(-6)-0(-3)^{10}(-4)^2(-8)}{2} = \boxed{-24}$

11. We being by noticing the coefficients of $3, 3, 1$, which remind us of the expansion of $(x+1)^3$. Hence, we apply the binomial theorem, yielding

$$9x^3 = x^3 + 3x^2 + 3x - 1$$

$$9x^3 = (x+1)^3$$

Now we have $\sqrt[3]{9}x = x + 1$, so $x = \frac{1}{\sqrt[3]{9}-1} = \frac{\sqrt[3]{81}+\sqrt[3]{9}+1}{8}$. We rationalize the denominator by multiplying the numerator and denominator by $\sqrt[3]{81} + \sqrt[3]{9} + 1$, which comes from the difference of cubes formula. Hence, $a + b + c = 81 + 9 + 8 = \boxed{098}$.

12. Let the roots be r_1 and r_2, with r_1 being the repeated root. Vieta's gives us $2r_1 + r_2 = -a$ and $r_1^2 r_2 = -9a$. Combining these gives $18r_1 + 9r_2 = r_1^2 r_2$, and isolating r_2 gives $r_2 = \frac{18r_1}{r_1^2-9}$. Since we know that the roots are integers, we must find integer solutions to this equation. By inspection, we can see that $\frac{18r_1}{r_1^2-9}$ cannot equal ± 1 for any integer value of r_1, and that $|r_1| \geq 10$ makes $|r_2| < 2$, which will produce no integer solutions. So, we must check the integers in the interval $[-9, 9]$ for r_1. The only nonzero integer solutions are $r_1 = 6$, $r_2 = 4$ and $r_1 = -6$, $r_2 = -4$. The corresponding values for a are ± 16. $b = r_1 r_1 + r_1 r_2 + r_1 r_2 = r_1^2 + 2r_1 r_2 = 84$, so $|ab| = |\pm 16 * 84| = \boxed{1344}$.

13. This proof uses an advanced technique known as Vieta jumping, which proceeds as follows.

- Suppose that $k = \frac{a^2+b^2}{ab+1}$ for some initial pair (a, b). Note that k must be positive. Without loss of generality, let $a \geq b$. Now we fix b and k, and we want to find value(s) of x that satisfy $\frac{x^2+b^2}{bx+1} = k$. Rearrange to get $x^2 - (kb)x + (b^2 - k) = 0$.
- One root of this equation is $x_1 = a$. Let the other root be x_2. By Vieta's formulas, x_2 may be written as $kb - a$ or as $\frac{b^2-k}{a}$.
- The first equation shows that x_2 is an integer. x_2 cannot be less than zero, because together with the fact that b is a positive integer it implies that $ab+1$ is less than or equal to 0, which would not allow k to be positive. $a \geq b$ implies that $x_2 = \frac{b^2-k}{a} < b$. Since the expression was symmetric, we now know that x_2 is a nonnegative integer less than b and that the pair (b, x_2) satisfies the equation.
- We essentially have a new pair (a_{new}, b_{new}). Using Vieta's we can repeatedly generate new pairs (a_n, b_n) that satisfy $\frac{a_n^2+b_n^2}{a_n b_n+1} = k$ and $b_n < b_{n-1}$ until b_z finally reaches 0 for some z. This means that k, which did not change during the entire process, is equal to $\frac{a_z^2+0}{0a_z+1} = a_z^2$, which is a perfect square. So, all values of k must be perfect squares.
- For an example of the root jumping process, note that $(112, 30)$ satisfies the equation $\frac{a^2+b^2}{ab+1} = 4$. The sequence is then $(112, 30), (30, 8), (8, 2), (2, 0)$.