

Lecture Notes: Combinatorics

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Combinatorics is the study of the “enumeration of discrete structures.” For high school math this is the study of counting and probability.

1. Counting

For distinguishable balls and distinguishable urns, power it up.

1.1. Stirling numbers of the second kind

For distinguishable balls and indistinguishable urns, use Stirling numbers of the second kind. $S(n, k)$ represents the number of ways to put n balls in exactly k urns.

$S(n + 1, k) = kS(n, k) + S(n, k - 1)$. Why?

1.2. Stars and Bars

For indistinguishable balls and distinguishable urns, use Stars and Bars.

1.3. Partitioning Integers

For indistinguishable balls and indistinguishable urns, use Partition. $p(n, k)$ represents the number of ways to partition the integer n into the sum of k nonincreasing integers. Most of the time, the simplest way is to just count them up.

However, there is the identity $p(n, k) = p(n - 1, k - 1) + p(n - k, k)$. Why?

So the number of ways to put 7 indistinguishable balls in 4 indistinguishable urns is $p(7, 1) + p(7, 2) + p(7, 3) + p(7, 4)$.

2. Combinatorial Identities

Identity 2.1 (Pascal’s Identity).

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Proof. (Proof by story.) Suppose you have a class of $n - 1$ students and 1 teacher. You want to select r of the n people to go on a field trip. You can either (a) not select the teacher and select r of the $n - 1$ students, or (b) select the teacher and select $r - 1$ of the $n - 1$ students. □

Identity 2.2.

$$\sum_{r=0}^n \binom{n}{r} = 2^n.$$

Proof. (Proof by story.) There are 2^n ways to pick a subset of n people (of any size) to go on a field trip. There are $\binom{n}{0}$ ways to choose 0 people, $\binom{n}{1}$ ways to choose 1 person, and so on, up to $\binom{n}{n}$ ways to choose n people. □

Identity 2.3 (Hockey Stick Identity).

$$\sum_{r=k}^n \binom{r}{k} = \binom{n+1}{k+1}.$$

Draw out the first few rows of Pascal's Triangle. What (geometrically) do the identities above tell you?

3. Recursion

3.1. Recursion

Use recursion.

Classic example: how many ways are there to cover a $2 \times n$ tile with 1×2 blocks?

4. Unrelated Stuff

4.1. Catalan Numbers

How many ways are there to get from $(0, 0)$ to (n, n) while remaining on or below the line $y = x$? Use the Catalan numbers!

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

4.2. Bertrand's Ballot Theorem

Theorem 4.1. *If you wish to go from $(0, 0)$ to (p, q) , where $p > q$, $\frac{p-q}{p+q}$ of all the possible paths will lie strictly under the line $y = x$ (besides at $(0, 0)$).*