Problem Set: Combinatorics

Montgomery Blair Math Team
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1. If I roll 3 fair 6-sided dice, what is the probability of me getting at least one 6?

2. a. In how many distinct ways can the letters of "cisforcookie" be ordered?
   b. In how many distinct ways can the letters of "cisforcookie" be ordered such that the letters the three o's show up after both of the c's?

3. Bob's Parking Lot is divided into the west lot and the east lot. There are 7 adjacent parking spaces in the west lot and 10 adjacent spaces in the east lot. If 13 identical cars and 2 identical trucks drive in and trucks require 2 adjacent parking spaces, how many ways are there to fill the lot?

4. How many four-digit base-20 integers have digits that sum to 13?

5. How many ways can you write a) 8 as a sum of 5 positive integers and b) 12 as a sum of 6 positive integers? Order does not matter.

6. Let $(a_1, a_2, a_3, ..., a_{12})$ be a permutation of $(1, 2, 3, ..., 12)$ for which $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and $a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$. An example of

such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations. (2006 AIME)

7. Consider all 1000-element subsets of the set 1, 2, 3, ... , 2015. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$. (2015 AIME)

   Hint: Hockey Stick Theorem.

8. Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers, where the kth layer from the top has 4^{k-1} blocks. Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters). (2010 HMMT)

9. How many numbers less than 361 have, in their prime factorizations, at least 2 primes less than 10?

10. Bob is handing out money to people who can beat his perfectly random strategy in rock-paper-scissors. Each contestant plays 10 rounds with Bob, and a round ends when one player wins (a round is redone if there is a tie). Let w be the number of wins that a contestant obtains. At the end of the 10 rounds, Bob gives $\$3^w$ to the contestant.

    a. What is the expected value of the number of games played, including ties? (That is, how many games would you play on average?)

b. What is the expected value of the amount of money won by a contestant?

11. Five runners are running in a race. If ties are allowed, how many possible end results can there be? (For example, one such result would be that runners A and D tie for first, B comes in next, and C and E tie for last.)

12. Sally is playing a game on the first quadrant and nonnegative x and y axes of the coordinate plane. Originally, there are 3 markers on the board: at $(0,0)$, $(1,0)$, and $(1,1)$. For each move, Sally may take a marker located at (x,y) off the board and replace it with markers at $(x+1,y)$, $(x+1,y+1)$, provided that these two spaces are empty. Prove that Sally cannot vacate $(0,0)$, $(1,0)$, and $(1,1)$.