

Lecture Notes: Introduction to Combinatorics

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1. The Basics

If you have n distinguishable objects, you can order them in $n! = 1 \cdot 2 \cdot \dots \cdot n$ ways. This is because you have n choices of which goes first; once you choose which one goes first, you have $n - 1$ choices of which goes second; and so on.

Now suppose the objects are not all distinguishable. How many ways are there to order them? (Hint: pretend they're distinguishable and then account for the ones that aren't distinguishable.)

Suppose you have 7 marbles, which you can tell apart, and you want to choose 3 of them. How many ways are there to do this? (Solution: you can select the first marble in 7 ways, the second marble in 6 ways, and the third marble in 5 ways. But the order in which you choose the three marbles is irrelevant, so the answer is $\frac{7 \cdot 6 \cdot 5}{3!} = 35$.)

Theorem 1.1. *The number of ways to choose r objects from a set of n distinguishable objects is*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

($\binom{n}{r}$ is pronounced “ n choose r ”.)

2. Stars and Bars

Suppose you want to put 10 balls in 5 cups. You can't tell the balls apart, but you can tell the cups apart. In how many ways can you do this if each cup must have a ball? (Answer: think of putting the balls into the 5 cups as having 4 bars and having to place each ball before the first bar (cup 1), between the first and second bar (cup 2), etc.) What about when you can have empty cups?

3. Combinatorial Identities

Identity 3.1 (Pascal's Identity).

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Proof. (Proof by story.) Suppose you have a class of $n - 1$ students and 1 teacher. You want to select r of the n people to go on a field trip. You can either (a) not select the teacher and select r of the $n - 1$ students, or (b) select the teacher and select $r - 1$ of the $n - 1$ students. \square

Identity 3.2.

$$\sum_{r=0}^n \binom{n}{r} = 2^n.$$

Proof. (Proof by story.) There are 2^n ways to pick a subset of n people (of any size) to go on a field trip. There are $\binom{n}{0}$ ways to choose 0 people, $\binom{n}{1}$ ways to choose 1 person, and so on, up to $\binom{n}{n}$ ways to choose n people. \square

Identity 3.3 (Hockey Stick Identity).

$$\sum_{r=k}^n \binom{r}{k} = \binom{n+1}{k+1}.$$

Draw out the first few rows of Pascal's Triangle. What (geometrically) do the identities above tell you? Try proving the identity using stars and bars.