Problem Set: Introduction to Combinatorics

Montgomery Blair Math Team

11/04/2015

1. How many three-element subsets of $\{1, 2, 3, 4, 5, 6\}$ exist? How many ordered triples consisting of three distinct elements of $\{1, 2, 3, 4, 5, 6\}$ exist? How many ordered triples consisting of three not-necessarily-distinct elements of $\{1, 2, 3, 4, 5, 6\}$ exist?

2. If I roll 3 fair 6-sided dice, what is the probability of me getting at least one 6?

3. a. In how many distinct ways can the letters of "cisforcookie" be ordered?
   b. In how many distinct ways can the letters of "cisforcookie" be ordered such that the letters the three o's show up after both of the c's?

4. Jefferson Davis has 12 identical flags and he wants to put the flags into three boxes A, B, and C. In how many ways can he do this? (If you want a hint, ask a history teacher.)

5. How many four-digit base-20 integers have digits that sum to 13?

6. In a school of 100 students, 40 students are enrolled in English, 40 students are enrolled in math, 25 students are enrolled in English and science, 20 students are enrolled in math and English, 10 students take all 3 classes, and every student takes at least one of these classes. What is the smallest possible number of students that could be enrolled in science?

7. How many ways can you write a) 8 as a sum of 5 positive integers and b) 12 as a sum of 6 positive integers? Order does not matter.

   Bonus: Let $f(n, k)$ be the number of ways to break integer n into k positive integers. Can you think of a recursion for $f(n, k)$?

8. A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers chose spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park? (2008 AMC12)

9. Let $(a_1, a_2, a_3, ..., a_{12})$ be a permutation of $(1, 2, 3, ..., 12)$ for which $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and $a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}$. An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations. (2006 AIME)

10. Consider all 1000-element subsets of the set 1, 2, 3, ... , 2015. From each such subset choose the least element. The arithmetic mean of all of these least elements is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$. (2015 AIME)

    Hint: Hockey Stick Theorem.

11. How many orderings $(a_1, ...a_8)$ of $(1, ..., 8)$ exist such that $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 = 0$? (2013 HMMT)

12. Bob is handing out money to people who can beat his perfectly random strategy in rock-paper-scissors. Each contestant plays 10 rounds with Bob, and a round ends when one player wins (a round is redone if there is a tie). Let w be the number of wins that a contestant obtains. At the end of the 10 rounds, Bob gives $\$3^w$ to the contestant.

    a. What is the expected value of the number of games played, including ties? (That is, how many games would you play on average?)

    b. What is the expected value of the amount of money won by a contestant?

13. Five runners are running in a race. If ties are allowed, how many possible end results can there be? (For example, one such result would be that runners A and D tie for first, B comes in next, and C and E tie for last.)