Solutions: Introduction to Combinatorics

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1. The first question asks how many ways there are to choose three things from a set of six things. This is $\binom{6}{3} = \frac{6!}{3!}3! = \boxed{20}$. (By the way, the best way to thing of $\binom{6}{3}$ is not as $\frac{6!}{3!}3!$ but as $\frac{6\cdot5\cdot4}{3\cdot2\cdot1}$.) For any triple you choose, there are 6 ways to order them, so the answer to the second question is $\boxed{120}$. For the third question, there are 6 ways to choose the first element of the triple, 6 ways to choose the second element, and 6 ways to choose the third element, for a total of $\boxed{216}$ choices.

2. The easiest way to do this problem is to count the number of ways to roll the dice so that you get no 6's, and then to subtract this number from the total number of ways to roll the dice, and then to divide this difference by the total number of ways to roll the dice. This technique is a common technique called complementary counting, where instead of counting the number of ways to do something, we count the number of ways that don't work, and subtract this from the total number of ways. Doing this, we get $\frac{6^3-5^3}{6^3} = \boxed{\dfrac{91}{216}}$.

3. a. There are 12 letters in the string, but we can't say that there are 12! ways to order the letters, because copies of the same letter are indistinguishable. Thus, we have do divide by 2! to account for the two c's, 2! to account for the i's, and 3! to account for the o's. Thus, our final answer is $\boxed{\dfrac{11!}{2}}$.

   b. The probability that any given ordering has the three o's showing up after both of the c's is $\frac{1}{\binom{5}{2}} = \frac{1}{10}$. Thus, we divide our previous answer by 10 to get $\boxed{\dfrac{11!}{20}}$.

   Alternatively, note that the answer is the number of ways that we can insert the non-c and non-o letters in and around "ccooo". This is essentially stars and bars with 5 bars and 7 stars, but we also have to take into consideration the number of ways to rearrange the seven letters. Thus, since we have two i's, there are $\frac{7!}{2}$ ways to rearrange the seven letters, so the answer is $\binom{12}{5}\frac{7!}{2} = \boxed{\dfrac{11!}{20}}$.

4. Suppose he puts the flags into three piles and has a bar between the piles. Each arrangement is equal to the number of ways to order 2 bars and 12 stars (hence Stars and Bars). Thus, our answer is $\binom{14}{2} = \frac{14\cdot13}{2} = \boxed{91}$.

5. We can look at each digit, and use the stars and bars method to split the sum of 13 into four digits. The first digit must be at least 1, so we allocate one star to that digit, which leaves us with 12 stars and 3 bars. Therefore, the answer is simply $\binom{15}{3} = \boxed{455}$

6. Let $A =$science, $B =$math, and $C =$English. $100 = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$, so $100 = A + 40 + 40 - |A \cap B| - 25 - 20 + 10$ or $A - |A \cap B| = 55$. $|A \cap B| \leq A$, $|A \cap B| \leq B = 40$, and $|A \cap B| \geq |A \cap B \cap C| = 10$.

Therefore, for A to be minimized, we must have $|A \cap B| = 10$ and consequently $A = \boxed{65}$.

7. Some careful counting gives $\boxed{3}$ for a) and $\boxed{11}$ for b).

   To find a recursion for $f(n,k)$, first note that a partition of n can either have a 1 in it, or it cannot have a 1 in it. These cases are mutually exclusive, so we can add the number of partitions from each case to get the answer. When we have 1 in our partition, note that we are essentially partitioning $n-1$ into $k-1$ partitions. When we do not have 1 in our partition, we can take then let us subtract 1 from each of our partitions. Then, the number of partitions in that case is equal to partitioning $n-k$ into k partitions. Therefore, $f(n,k) = f(n-1,k-1) + f(n-k,k)$.

8. The probability that Auntie Em can park is equal to 1 minus the probability that she can't park. She only cannot park when all four open spots are separated. We can select 4 spots out of 13 spots and then insert an extra car between adjacent spots. **_*_***_*_** would become **_**_****_**_**, for example. This means that in total there are $\binom{13}{4}$ ways that she cannot park, for a probability of $1-\binom{13}{4}/\binom{16}{4} = \boxed{\frac{17}{28}}$.

9. $a_1, \ldots, a_5$ and $a_7, \ldots, a_{12}$ are all greater than a_6 so a_6 must be 1. We can select any set of 5 unique numbers from $\{2, \ldots, 12\}$ to be $a_1, \ldots, a_5$ once sorted in descending order. The remaining 6 numbers can be $a_7, \ldots, a_{12}$ once sorted in descending order. Therefore, there are $\binom{11}{5} = \binom{11}{6} = \boxed{462}$ tuples $(a_1, \ldots, a_{12})$.

10. Note that there are $\binom{2015}{1000}$ subsets, and that there are $\binom{2015-k}{999}$ subsets with smallest element k. Therefore, the mean is

$$\frac{\sum_{n=1}^{1016} n\binom{2015-n}{999}}{\binom{2015}{1000}} = \frac{\sum_{m=1}^{1016}\sum_{n=1}^{m}\binom{2015-n}{999}}{\binom{2015}{1000}} = \frac{\sum_{m=1}^{1016}\binom{2016-m}{1000}}{\binom{2015}{1000}} = \frac{\binom{2016}{1001}}{\binom{2015}{1000}} = \frac{2016}{1001} = \frac{288}{143}$$

    Therefore, the answer is $\boxed{431}$.

11. We can partition $(1, \ldots, 8)$ into two equal valued sets of four elements four different ways ($\{1,2,7,8\}$ and $\{3,4,5,6\}$, $\{1,3,6,8\}$ and $\{2,4,5,7\}$, $\{1,4,5,8\}$ and $\{2,3,6,7\}$, and $\{1,4,6,7\}$ and $\{2,3,5,8\}$). For each of the four partitions, we can choose which of the two sets is added or subtracted and then assign all permutations of the sets' elements to a_1, a_3, a_5, a_7 and a_2, a_4, a_6, a_8, respectively. This yields $4 \times 2 \times 4! \times 4! = \boxed{4608}$.

12. a. The expected value is $10(1 + \frac{1}{3} + \frac{1}{9} + \ldots)$. The sum is a geometric series equalling $\frac{3}{2}$, making the total expected value $\boxed{15}$.

    b. The probability of winning any round is clearly $\frac{1}{2}$. Consider the polynomial formed by the expansion of $(\frac{1}{2}x + \frac{1}{2})^{10}$. Each coefficient of x^n represents the probability of obtaining n wins. The total expected value of money gained can be obtained by simply plugging 3 in for x in the above polynomial, giving $2^{10} = \boxed{1024}$.

13. See here for the general approach. With this approach we get that the final answer is $\boxed{541}$