Problem Set: Introduction to Number Theory

Montgomery Blair Math Team
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1. Commit the following prime factorizations to memory; they will almost certainly be useful to you in math competitions (and in some of the problems in this problem set): $1001 = 7 \cdot 11 \cdot 13$; $2014 = 2 \cdot 19 \cdot 53$; $2015 = 5 \cdot 13 \cdot 31$; $2016 = 2^5 \cdot 3^2 \cdot 7$; 2017 is prime.

2. (a) Use the Euclidean algorithm to find the $\gcd(98, 238)$.

   (b) Find the least common multiple of 98 and 238.

   (c) Prove that the product of your answers in parts (a) and (b) is equal to $98 \cdot 238$.

3. What is the remainder when 11^{12} is divided by 13? (MathCounts state sprint 2014)

4. How many zeroes are after the last nonzero digit of $2016_{10}!$ in base 12?

5. What are all positive integers n such that $\text{lcm}(2n, n^2) = 14n - 24$? (PUMaC 2015)

6. How many positive integers less than or equal to 2015 are relatively prime to 2015? How many positive integers less than or equal to 100 are relatively prime to 100?

7. What is the smallest integer n greater than 2011 such that $n-1 \equiv (n-1)! \pmod{n}$?

8. What is the remainder when 17^{1442} is divided by 1001?

9. Find the inverse of 17 (mod 90) (that is, the number x such that $0 \leq x < 90$ and $17x \equiv 1 \pmod{90}$). (Hint: use the extended Euclidean algorithm.)

10. What is the smallest positive integer n such that $2^n - 1$ is a multiple of 2015? (PUMaC 2015)

11. Find the number of rational numbers r, $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and denominator have a sum of 1000. (AIME I 2014)

12. A grocer orders apples and oranges at a total cost of \$8.39. If apples cost 25 cents each and oranges cost 18 cents each, how much of each type of fruit did the grocer order? (University of Western Australia)

13. The numbers in the sequence $100, 101, 104, 109, 116, \ldots$ are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$. For each n, let d_n be the greatest common factor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers. (AIME 1985)

14. Let d be the greatest common divisor of $2^{30^{10}} - 2$ and $2^{30^{45}} - 2$. Find the remainder when d is divided by 2013. (PUMaC 2013)

15. Let a,b be integers chosen independently and uniformly at random from the set $\{0, 1, 2, \ldots, 80\}$. Compute the expected value of the remainder when $\binom{a}{b}$ is divided by 3. Note that $\binom{0}{0} = 1$ and $\binom{a}{b} = 0$ when $a < b$. (HMMT 2015)

16. Find, with proof, all integers n for which $2^n + 12^n + 2011^n$ is a perfect square. (USAJMO 2011)