Solutions: Introduction to Number Theory

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1. N/A.

2. (a) $\gcd(98, 238) = \gcd(98, 42) = \gcd(14, 42) = \gcd(14, 14) = \boxed{14}$.

   (b) 98 and 238 have 14 in common. $98 = 14 \cdot 7$ and $238 = 14 \cdot 17$. 7 and 17 are relatively prime (this would be the case no matter what the original two numbers were, since otherwise the GCD would be different to account for the common factor). Thus, the answer is $14 \cdot 7 \cdot 17 = \boxed{1666}$.

   (c) We only included the greatest common factor once when we found the least common multiple, whereas each of the two numbers has the greatest common factor once. Thus, to find $98 \cdot 238$, we must multiply 1666 by 14.

3. By Fermat's Little Theorem, the answer is $\boxed{1}$.

4. We wish to find the number of factors of 12 in 2016!. There are $\lfloor \frac{2016}{2} \rfloor + \lfloor \frac{2016}{4} \rfloor + \lfloor \frac{2016}{8} \rfloor + \lfloor \frac{2016}{16} \rfloor + \lfloor \frac{2016}{32} \rfloor + \lfloor \frac{2016}{64} \rfloor + \lfloor \frac{2016}{128} \rfloor + \lfloor \frac{2016}{256} \rfloor + \lfloor \frac{2016}{1024} \rfloor = 2010$ factors of 2 in 2016!, and $\lfloor \frac{2016}{3} \rfloor + \lfloor \frac{2016}{9} \rfloor + \lfloor \frac{2016}{27} \rfloor + \lfloor \frac{2016}{81} \rfloor + \lfloor \frac{2016}{243} \rfloor + \lfloor \frac{2016}{729} \rfloor = 1004$ factors of 3 in 2016!. Since $12 = 2^2 \cdot 3$, there are $\boxed{1004}$ factors of 12 in 2016!.

5. Since $2015 = 5 \cdot 13 \cdot 31$ (as you should know), we have $\phi(2015) = 2015 \cdot \frac{4}{5} \cdot \frac{12}{13} \cdot \frac{30}{31} = \boxed{1440}$.

   Since $100 = 2^2 \cdot 5^2$, we have $\phi(100) = 100 \cdot \frac{1}{2} \cdot \frac{4}{5} = \boxed{40}$.

6. See #2 here.

7. We want to find the smallest $n > 2011$ such that $-1 \equiv (n-1)! \pmod{n}$. By Lucas' Theorem, this is true iff n is prime. The smallest prime larger than 2011 is $\boxed{2017}$.

8. As you should know, $1001 = 7 \cdot 11 \cdot 13$, so $\phi(1001) = 1001 \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} = 720$. By the Euler-Fermat Theorem (since 17 is relatively prime to 1001), we have $17^{720} \equiv 1 \pmod{1001}$, so $17^{1440} \equiv 1 \pmod{1001}$. Thus, $17^{1442} \equiv 17^2 \equiv \boxed{289} \pmod{1001}$.

9. $17x \equiv 1 \pmod{90}$ is the same as writing $17x + 90y = 1$. Since 17 and 90 are relatively prime, $17x + 90y = 1 = \gcd 1790$, which allows for the use of the extended Euclidean algorithm to solve for x (and y is irrelevant). Applying the algorithm, you find that $x = \boxed{53}$.

10. See #6 here.

11. We want to count the number of $\frac{a}{b}$ where $a + b = 1000$, $a < b$, and $\frac{a}{b}$ is in simplest form. Substituting, $\frac{1000-b}{b} = \frac{1000}{b} - 1$, so $\frac{1000}{b}$ must also be in simplest form. Since b ranges from 501 to 999, there are 249 values of b that are divisible by 2, 99 that are divisible by 5, and 49 that are divisible by 2 and 5. By Inclusion/Exclusion, $499 - 249 - 99 + 49 = \boxed{200}$.

12. We wish to find a and b such that $839 = 25a + 18o$. Applying the extended Euclidean algorithm to $1 = \gcd 1825 = 25x + 18y$, we find that $1 = 25(-5) + 18(7)$. Multiplying by 839, $839 = 25(-4195) + 18(5873)$. Since $0 = 25(18) - 18(25)$, we can say that $839 = 25(-4195 + 18z) + 18(5873 - 25z) = 25(17) + 18(23)$. Therefore, $\boxed{a = 17 \text{ and } o = 23}$.

13. See here.

14. See #4 here.

15. See #27 here.

16. See here.