Problem Set: Sequences and Series

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1. One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. The ratio of the total area of the green regions to the area of the circle of radius 100 can be expressed as m/n, where m and n are relatively prime positive integers. Find $m+n$.

2. An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2,$ Find the possible value(s) of a_{2007}.

3. Let $x_1 = \sqrt{10}$ and $y_1 = \sqrt{3}$. For all $n \geq 2$, let

$$x_n = x_{n-1}\sqrt{77} + 15y_{n-1}$$
$$y_n = 5x_{n-1} + y_{n-1}\sqrt{77}$$

Find $x_5^6 + 2x_5^4 - 9x_5^4y_5^2 - 12x_5^2y_5^2 + 27x_5^2y_5^4 + 18y_5^4 - 27y_5^6$.

4. Let $a_1, a_2, ...$ be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find

$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}$$

5. Compute the value of the infinite series:

$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n(n^4 + 4)}$$

6. Compute

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}$$