Problem Set: Sequences and Series

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1. One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. The ratio of the total area of the green regions to the area of the circle of radius 100 can be expressed as m/n, where m and n are relatively prime positive integers. Find $m + n$.

   Solution: To get the green area, we can color all the circles of radius 100 or below green, then color all those with radius 99 or below red, then color all those with radius 98 or below green, and so forth. This amounts to adding the area of the circle of radius 100, but subtracting the circle of radius 99, then adding the circle of radius 98, and so forth.

   The total green area is thus given by $100^2\pi - 99^2\pi + 98^2\pi - \ldots - 1^2\pi$, while the total area is given by $100^2\pi$, so the ratio is
$$\frac{100^2\pi - 99^2\pi + 98^2\pi - \ldots - 1^2\pi}{100^2\pi}$$
   For any a, $a^2 - (a-1)^2 = (a + (a-1))(a - (a-1)) = 2a - 1$. We can cancel the factor of pi from the numerator and denominator and simplify the ratio to
$$\frac{(2\cdot 100 - 1) + (2\cdot 98 - 1) + \ldots + (2\cdot 2 - 1)}{100^2} = \frac{2\cdot(100 + 98 + \ldots + 2) - 50}{100^2}.$$
   Using the formula for the sum of an arithmetic series, we see that this is equal to
$$\frac{2(50)(51) - 50}{100^2} = \frac{50(101)}{100^2} = \frac{101}{200},$$
   so the answer is $101 + 200 = \boxed{301}$.

2. An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2,$ Find the possible value(s) of a_{2007}.

   Solution: The characteristic equation of the recurrence relation is: $\lambda^2 + \lambda - 6 = 0 \iff (\lambda+3)(\lambda - 2) = 0 \iff \lambda = -3 \quad \text{or} \quad \lambda = 2$

   Thus the general solution is $a_n = c_1(-3)^n + c_2 2^n \quad n \in \mathbb{N} \cup \{0\}$ where c_1, c_2 are real constants.

   If $c_1 \neq 0$ then for large enough values of n we get negative terms of the sequence which counters the hypothesis since $a_n > 0$ for all n.

   So $c_1 = 0$ and we find $a_n = c_2 2^n$and using the initial condition $a_0 = 1$ we find out that $c_2 = 1$.

   Finally, $a_n = 2^n$.

   Thus, $\boxed{a_{2007} = 2^{2007}}$

3. Let $x_1 = \sqrt{10}$ and $y_1 = \sqrt{3}$. For all $n \geq 2$, let
$$x_n = x_{n-1}\sqrt{77} + 15y_{n-1}$$
$$y_n = 5x_{n-1} + y_{n-1}\sqrt{77}$$
   Find $x_5^6 + 2x_5^4 - 9x_5^4y_5^2 - 12x_5^2y_5^2 + 27x_5^2y_5^4 + 18y_5^4 - 27y_5^6$.

Solution: We have that $x_5^6 - 9x_5^4y_5^2 + 27x_5^2y_5^4 - 27y_5^6 = (x_5^2 - 3y_5^2)^3$ and also we have $2x_5^4 - 12x_5y_5^2 + 18y_5^4 = 2(x_5^2 - 3y_5^2)^2$

At first this inspired me to look at $x_5^2 - 3y_5^2$ but for this solution I'll generalize.

We have that

$$\begin{aligned} x_n^2 - 3y_n^2 &= 77x_{n-1}^2 + 30\sqrt{77}x_{n-1}y_{n-1} + 225y_{n-1}^2 - 75x_{n-1}^2 - 30\sqrt{77}x_{n-1}y_{n-1} - 231y_{n-1}^2 \\ &= 2x_{n-1}^2 - 6y_{n-1}^2 \\ &= 2(x_{n-1}^2 - 3y_{n-1}^2) \end{aligned}$$

It follows that

$$x_5^2 - 3y_5^2 = 2(x_4^2 - 3y_4^2) = 4(x_3^2 - 3y_3^2) = 8(x_2^2 - 3y_2^2) = 16(x_1^2 - 3y_1^2) = 16(10 - 9) = 16$$

Thus we have that the expression is equal to $(x_5^2 - 3y_5^2)^3 + 2(x_5^2 - 3y_5^2)^2 = 16^3 + 2 * 16^2 = 2^{12} + 2^9 = 4096 + 512 = \boxed{4608}$

4. Let $a_1, a_2, ...$ be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find

$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}$$

Solution: Look at the generating function for the Fibonacci numbers: $\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + \ldots + F_nx^n + \ldots$. Plug in $x = \frac{1}{4}$ to get $\frac{\frac{1}{4}}{1-\frac{1}{4}-\frac{1}{16}} = \frac{4}{16-4-1} = \frac{4}{11} = \sum_{n=1}^{\infty} \frac{F_n}{4^n}$. Dividing by 4 gives the answer of $\frac{1}{11}$.

5. Compute the value of the infinite series:

$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n(n^4 + 4)}$$

Solution: $\frac{n^4+3n^2+10n+10}{2^n \cdot (n^4+4)} = \frac{1}{2^n}(1 + \frac{4}{n^2-2n+2} - \frac{1}{n^2+2n+2}) = \frac{1}{2^n} + \frac{1}{2^n}(\frac{4}{n^2-2n+2} - \frac{1}{n^2+2n+2})$

$$\sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}$$

$$\sum_{n=2}^{\infty} \frac{1}{2^n}\left(\frac{4}{n^2-2n+2} - \frac{1}{n^2+2n+2}\right) = \sum_{n=2}^{\infty} \frac{1}{2^n}\left(\frac{4}{(n-1)^2+1} - \frac{1}{(n+1)^2+1}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n}\left(\frac{1}{(n+1)^2+1}\right) - \sum_{n=2}^{\infty} \frac{1}{2^n}\left(\frac{1}{(n+1)^2+1}\right) = \frac{6}{10}$$

Therefore, the answer is $\boxed{\frac{11}{10}}$

6. Compute

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}$$

Solution: This is Problem 3 on the 2012 Edition of PUMaC Algebra. You will find the solution with this link: 2012 PUMaC Algebra A Solutions

You can also break $\frac{n+1}{n^2(n+2)^2}$ into $\frac{1}{4}(\frac{1}{n^2} - \frac{1}{(n+2)^2})$ and use a telescoping series.